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A Note on the Rate of Convergence of Sturm–Liouville Expansions

ZITA DIVIS

Department of Mathematics, Ohio State University, Columbus, Ohio 43210, U.S.A.

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We give an extension of Jackson's theorem on the rate of convergence of eigenfunction expansions of Lip 1 functions to functions of bounded variation.

1. INTRODUCTION

Let q be a continuous function on $[0, \pi]$ and (ϕ_n) the sequence of orthonormalized eigenfunctions of the Sturm–Liouville problem

$$y'' + (\lambda^2 - q(x))\, y = 0 \tag{1}$$

$$\begin{aligned} y'(0) - hy(0) &= 0 \\ y'(\pi) + Hy(\pi) &= 0 \end{aligned} \tag{2}$$

where h, H are real numbers. Let f be an integrable function on $[0, \pi]$ and consider the eigenfunction expansion of f

$$\sum_{0}^{\infty} a_k \phi_k(x) \tag{3}$$

where $a_k = \int_0^\pi f(t)\, \phi_k(t)\, dt$.

It is wel known [T] that the series (3) is convergent whenever f is a function of bounded variation on $[0, \pi]$ and its sum is $\frac{1}{2}(f(x+0) + f(x-0))$ for every $x \in (0, \pi)$.

Estimates for the rate of convergence of (3) were given by Jackson [J] under various hypotheses, mostly requiring some differentiability of f and

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q. We shall cite here Jackson's result where no differentiability assumptions are made:

If f satisfies the Lipschitz condition

$$|f(x_1)-f(x_2)| \leqslant \mu |x_1 - x_2|$$

on $[0, \pi]$ *and if* $f(0)=0$ *and q is continuous on* $[0, \pi]$ *then for all* $x \in [0, \pi]$

$$|f(x) - S_n(f, x)| \leqslant \frac{c\mu \ln n}{n}, \qquad n \geqslant 2, \tag{4}$$

where $S_n(f, x) = \sum_{k=0}^{n} a_k \phi_k(x)$ *and c is a constant independent of* x, f, n.

Bojanic and Divis [BD] obtained the following result:

If f is a function of bounded variation on $[0, \pi]$ *and q is continuous on* $[0, \pi]$ *then for* $x \in (0, \pi)$ *and all n sufficiently large we have*

$$\left|\frac{1}{2}(f(x+0)+f(x-0)) - S_n(f, x)\right| \leqslant \frac{M(f, x)}{n} \cdot \sum_{k=1}^{n} V_{x-x/k}^{x+(\pi-x)/k}(g_x). \tag{5}$$

Here, $S_n(f, x)$ is as in the above result, M is a positive quantity depending only on f and x,

$$\begin{aligned} g_x(t) &= f(t) - f(x-0), && 0 \leqslant t < x, \\ &= 0, && t = x, \\ &= f(t) - f(x+0), && x < t \leqslant \pi, \end{aligned}$$

and $V_a^b(g)$ is the total variation of g on $[a, b]$.

Remark. This theorem is not a direct extension of Jackson's result. For f in Lip 1 class, we have, namely,

$$V_{x-x/k}^{x+(\pi-x)/k}(g_x) \leqslant \frac{c}{k},$$

but M still depends on x in the estimate (5).

The aim of this paper is to show that an estimate similar to (5) is available for functions of bounded variation on $[0, \pi]$ where the bound $M(f, x)$ is replaced by a bound independent of x. The present approach is more elementary and uses only properties of asymptotic expansion of eigenfunctions and eigenvalues.

The proof of the main result is based on the technique used by Jackson [J] and the following theorem of Bojanic [B] for Fourier series:

If f is a 2π-periodic function of bounded variation on $[-\pi, \pi]$ and $\sigma_n(f, x)$ is the nth partial sum of the Fourier expansion of f then for all $n \geq 1$

$$\left|\frac{1}{2}(f(x+0)+f(x-0))-\sigma_n(f, x)\right| \leq \frac{3}{n} \sum_{k=1}^{n} V_0^{\pi/k}(\tilde{g}_x) \qquad (6)$$

where $\tilde{g}_x(t) = f(x+t) + f(x-t) - f(x+0) - f(x-0)$.

2. Formulation and Proof of the Result

THEOREM. *Let f be a function of bounded variation on $[0, \pi]$ and let q be continuous on $[0, \pi]$. If we denote $M(f) = |f(0)| + V_0^{\pi}(f)$ then for all $n \geq 1$ and $x \in (0, \pi)$ we have*

$$\left|\frac{1}{2}(f(x+0)+f(x-0))-S_n(f, x)\right| \leq \frac{c}{n}\left(M(f) + \sum_{k=1}^{n} V_0^{\pi/k}(\tilde{g}_x)\right)$$

where $\tilde{g}_x$ is as in the above therem and c is independent of x, n, f.

In the special case when f is Lip 1, *we have $V_0^{\pi/k}(\tilde{g}_x) \leq c/k$ and so this result clearly extends Jackson's result to functions of bounded variation.*

Before we proceed to prove the theorem let us recall some well-known facts about Sturm–Liouville expansions. First, eigenvalues λ_i^2 of the problem (1), (2) are real and simple and we can write $\lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \cdots$. We have then

$$\lambda_n = n + \varepsilon_n \qquad (7)$$

where $\varepsilon_n = O(1/n)$ (see, e.g., [T] or [J]). For each λ_i^2 there is only one (up to a constant factor) eigenfunction of (1), (2). Next, if ψ_n is an eigenfunction of (1), (2) then the constant factor can be chosen in such a way that ψ_n satisfies the integral equation (for $\lambda_n^2 > 0$)

$$\psi_n(x) = \cos \lambda_n x + \frac{h}{\lambda_n} \sin \lambda_n x + \frac{1}{\lambda_n} \int_0^x q(t)\, \psi_n(t) \sin \lambda_n(x-t)\, dt \qquad (8)$$

(see [J] or [T]). An immediate consequence of this integral equation is the asymptotic relation

$$\psi_n(x) = \cos \lambda_n x + O(1/n) \qquad (9)$$

valid uniformly for $x \in [0, \pi]$. This shows, in particular, that eigenfunctions are uniformly bounded on $[0, \pi]$.

We want to study the rate of convergence of the series $\sum_0^\infty a_k \phi_k(x)$. However, it will be more convenient to consider instead the series $\sum_0^\infty A_k \psi_k(x)$ where ψ_k are the eigenfunctions satisfying the integral equation (8) and then, of course,

$$A_k = (\|\psi_k\|_2^2)^{-1} \int_0^\pi f(t)\, \psi_k(t)\, dt. \tag{10}$$

We shall now investigate the behaviour of the coefficients A_n. For the denominator, a simple relation given by Liouville (see, e.g., [J]) is sufficient

$$\|\psi_n\|_2^2 = \pi/2 + O(1/n) \tag{11}$$

and thus we shall turn our attention to the numerator of A_n using the integral equation (8) satisfied by the functions ψ_n. We have then

$$\int_0^\pi f(x)\, \psi_n(x)\, dx = \int_0^\pi f(x) \cos \lambda_n x\, dx + \frac{h}{\lambda_n} \int_0^\pi f(x) \sin \lambda_n x\, dx$$
$$+ \frac{1}{\lambda_n} \int_0^\pi f(x) \left(\int_0^x q(t)\, \psi_n(t) \sin \lambda_n (x - t)\, dt \right) dx. \tag{12}$$

Our aim is to use (12) to show that

$$A_n = \int_0^\pi \frac{f(t)\, \psi_n(t)}{\|\psi_n\|_2^2}\, dt = \frac{2}{\pi} \int_0^\pi f(t) \cos nt\, dt + O\left(\frac{1}{n^2}\right).$$

In what follows c will always denote a constant independent of x, f, n. Of course, c depends on q, h, H. Note that for $x \in [0, \pi]$ we have $|f(x)| \leqslant M(f) \equiv |f(0)| + V_0^\pi(f)$.

LEMMA 1. *If f is of bounded variation on $[0, \pi]$ and $\lambda_n = n + O(1/n)$ then for any $0 \leqslant a \leqslant b \leqslant \pi$ we have*

$$\left| \int_a^b f(x) \sin \lambda_n x\, dx \right| \leqslant \frac{cM(f)}{n}$$

and

$$\left| \int_a^b f(x) \cos \lambda_n x\, dx \right| \leqslant \frac{cM(f)}{n}$$

for all $n = 1, 2, \ldots$.

Proof follows using integration by parts.

LEMMA 2. *If f is of bounded variation, q continuous on $[0, \pi]$, and $\lambda_n = n + O(1/n)$, then for all $n = 1, 2,...$ we have*

$$\left| \int_0^\pi f(x) \left(\int_0^x q(t)\, \psi_n(t) \sin \lambda_n(x-t)\, dt \right) dx \right| \leqslant \frac{cM(f)}{n}.$$

Proof. We shall first reverse the order of integration. Then the integral in question equals

$$\int_0^\pi q(t)\, \psi_n(t) \left(\int_t^\pi f(x) \sin \lambda_n(x-t)\, dx \right) dt$$

$$= \int_0^\pi q(t)\, \psi_n(t) \cos \lambda_n t \left(\int_t^\pi f(x) \sin \lambda_n x\, dx \right) dt$$

$$- \int_0^\pi q(t)\, \psi_n(t) \sin \lambda_n t \left(\int_t^\pi f(x) \cos \lambda_n x\, dx \right) dt.$$

We can now estimate both $\int_t^\pi f(x) \sin \lambda_n x\, dx$ and $\int_t^\pi f(x) \cos \lambda_n x\, dx$ in view of Lemma 1 by

$$cM(f)/n \qquad \text{for} \quad n \geqslant 1.$$

Consequently, taking into account (9), the desired inequality follows.

As a consequence of Lemmas 1 and 2 and the relation (12) we now have

$$\int_0^\pi f(x)\, \psi_n(x)\, dx = \int_0^\pi f(x) \cos \lambda_n x\, dx + O\left(\frac{1}{n^2} \right). \tag{13}$$

LEMMA 3. *If f is of bounded variation on $[0, \pi]$ and $\lambda_n = n + O(1/n)$ we have the following estimate for all $n \geqslant 1$:*

$$\left| \int_0^\pi f(x) \cos \lambda_n x\, dx - \int_0^\pi f(x) \cos nx\, dx \right| \leqslant \frac{cM(f)}{n^2}.$$

Proof. Writing $\lambda_n = n + \varepsilon_n$ with $\varepsilon_n = O(1/n)$ we obtain

$$\left| \int_0^\pi f(x) \cos(n + \varepsilon_n)\, x\, dx - \int_0^\pi f(x) \cos nx\, dx \right|$$

$$\leqslant \left| \int_0^\pi f(x) \cos nx (\cos \varepsilon_n x - 1)\, dx \right| + \left| \int_0^\pi f(x) \sin \varepsilon_n x \sin nx\, dx \right|.$$

Now, in the first integral, $\cos \varepsilon_n x - 1 = O(1/n^2)$. (Estimate is independent of $x \in [0, \pi]$.) In the second integral, the product $f(x) \sin \varepsilon_n x$ has bounded variation on $[0, \pi]$ and

$$\begin{aligned} V_0^\pi(f(x) \sin \varepsilon_n x) &\leqslant \sup_{[0,\pi]} |f(x)| \cdot V_0^\pi(\sin \varepsilon_n x) \\ &\quad + \sup_{[0,\pi]} |\sin \varepsilon_n x| \cdot V_0^\pi(f) \\ &\leqslant \frac{cM(f)}{n} \qquad \text{for all} \quad n \geqslant 1 \end{aligned}$$

so that

$$\begin{aligned} \left| \int_0^\pi f(x) \sin \varepsilon_n x \, d\left(-\frac{\cos nx}{n}\right) \right| &= \left| -\frac{1}{n} f(x) \sin \varepsilon_n x \cos nx \right|_0^\pi \\ &\quad \left. + \frac{1}{n} \int_0^\pi \cos nx \, d(f(x) \sin \varepsilon_n x) \right| \\ &\leqslant \frac{cM(f)}{n^2} \end{aligned}$$

and the lemma is established. In view of Lemma 3 the relation (13) becomes

$$\int_0^\pi f(x)\, \psi_n(x)\, dx = \int_0^\pi f(x) \cos nx \, dx + O\left(\frac{1}{n^2}\right),$$

or more precisely

$$\left| \int_0^\pi f(x)\, \psi_n(x)\, dx - \int_0^\pi f(x) \cos nx \, dx \right| \leqslant \frac{cM(f)}{n^2} \quad \text{for all } n \geqslant 1. \tag{14}$$

Next we shall consider an even 2π-periodic extension of f and its cosine Fourier expansion $\sum_0^\infty \alpha_k \cos kx$. Recall that the series we are investigating is $\sum_0^\infty A_k \psi_k(x)$ with $A_k = (\|\psi_k\|_2^2)^{-1} \int_0^\pi f(x)\, \psi_k(x)\, dx$. We shall now prove the following

LEMMA 4. *If f has bounded variation on $[0, \pi]$ then*

$$|A_n \psi_n(x) - \alpha_n \cos nx| \leqslant \frac{cM(f)}{n^2} \qquad \textit{for all} \quad n \geqslant 1.$$

Proof. Consider the expression

$$A_n\psi_n(x) = (\|\psi_n\|_2^2)^{-1}\,\psi_n(x)\int_0^\pi f(t)\,\psi_n(t)\,dt.$$

In this product, we shall use the Liouville relation (11) for $\|\psi_n\|_2^2$, estimate (14) for $\int_0^\pi f(t)\,\psi_n(t)\,dt$, and the relation

$$\psi_n(x) = \cos nx + O(1/n)$$

which is a simple consequence of (9) and valid uniformly in $x \in [0, \pi]$. It follows then that

$$A_n\psi_n(x) = \left(\frac{2}{\pi} + O\left(\frac{1}{n}\right)\right)\left(\cos nx + O\left(\frac{1}{n}\right)\right)\left[\int_0^\pi f(t)\cos nt\,dt + R_n\right]$$

where $|R_n| \leqslant cM(f)/n^2$. Taking into account that also $|\int_0^\pi f(t)\cos nt\,dt| \leqslant cM(f)/n$ we finally obtain

$$A_n\psi_n(x) = \frac{2}{\pi}\int_0^\pi f(x)\cos nt\,dt + R_n'$$

with $|R_n'| \leqslant cM(f)/n^2$ and the lemma is proved.

Proof of the Theorem. Recall that $S_n(f, x) = \sum_{k=0}^n a_k\phi_k(x) = \sum_{k=0}^n A_k\psi_k(x)$ and let us denote $\sigma_n(f, x) = \sum_{k=0}^n \alpha_k \cos kx$. Consider now the difference

$$\begin{aligned}
&\left|\frac{1}{2}(f(x+0)+f(x-0)) - S_n(f, x)\right| \leqslant \\
&\quad\left|\frac{1}{2}(f(x+0)+f(x-0)) - \sigma_n(f, x)\right| \\
&\qquad + \left|\left[\frac{1}{2}(f(x+0)+f(x-0)) - S_n(f, x)\right]\right. \\
&\qquad \left. - \left[\frac{1}{2}(f(x+0)+f(x-0)) - \sigma_n(f, x)\right]\right| \\
&\quad = \left|\frac{1}{2}(f(x+0)+f(x-0)) - \sigma_n(f, x)\right| \\
&\qquad + \left|\sum_{k=n+1}^{\infty}[A_k\psi_k(x) - \alpha_k\cos kx]\right|.
\end{aligned}$$

We shall now use estimate (6) for Fourier series and Lemma 4 to obtain for all $n \geqslant 1$ and all $x \in (0, \pi)$

$$\left|\frac{1}{2}(f(x+0)+f(x-0))-S_n(f, x)\right| \leqslant \frac{3}{n} \sum_{k=1}^{n} V_0^{\pi/k}(\tilde{g}_x)+\sum_{k=n+1}^{\infty} \frac{cM(f)}{k^2}$$
$$\leqslant \frac{c}{n}\left(M(f)+\sum_{k=1}^{n} V_0^{\pi/k}(\tilde{g}_x)\right)$$

and the theorem is proved.

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